

Vertex Removable Cycles of Graphs and Digraphs

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Abstract. In this paper we defined the vertex removable cycle that, if \mathfrak{S} is a class of graphs(digraphs) satisfying certain property, $G \in \mathfrak{S}$, the cycle C in G is called vertex removable if $G - V(C) \in \mathfrak{S}$. The vertex removable cycles of eulerian graphs have been studied. Also we characterized the vertex removable cycles for regular graphs(digraphs).

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1. Introduction

Borse and B.N.Waphare [2] defined a non-separating cycle that a cycle C in a graph G is called non-separating if $G - V(C)$ is connected. Attar [1] defined the edge removable cycle that, if \mathfrak{S} be a class of graphs (digraphs), satisfying some property, the cycle C in is called edge removable if $G - E(C) \in \mathfrak{S}$.

We defined the vertex removable cycle as follows:

Definition 1.1. *Let \mathfrak{S} be a class of graphs (digraphs) satisfying certain property, $G \in \mathfrak{S}$, the cycle C in G is called vertex removable if $G - E(C) \in \mathfrak{S}$.*

Thomassen and Toft [7] proved that a connected graph of minimum degree at least 3 contains an induced non-separating cycle.

It follows from Thomassen result that the connected graph of minimum degree at least 3 contains a vertex removable cycle.

Theorem 1.2. *Let G be a connected graph of minimum degree at least 3. Then there exist vertex disjoint induced cycles C_1 and C_2 in G such that both $G - V(C_1)$ and $G - V(C_2)$ are connected if and only if $|V(G)| \geq 6$ and is not isomorphic to any of the following graphs: (i) a wheel; (ii) a graph with a partition (X, Y) of its vertex set such that $|X| = 3$ and $N(y) = X$ for all $y \in Y$.*

Tutte [8] proved a special case of a conjecture due to Lovasz [6] by proving that there exists an induced non-separating cycle in a 3-connected graph through a given edge xy and avoiding a given vertex z with $z \notin x, y$. Borse [2] strengthen this result.

Theorem 1.3. *Let G be a 3-connected graph and let H be a connected subgraph of G and $vw \in E(G - V(H))$. Suppose that there exists a cycle containing vw in $G - V(H)$. Then there exists an induced cycle C containing vw in $G - V(H)$ such that $G - V(C)$ is connected.*

Further, He obtained the following theorem for non-separating even cycles.

Theorem 1.4. *Let G be a connected graph and let H be a connected subgraph of G such that $G - V(H)$ contains an even cycle. Suppose that $d_G(v) \geq 4$ for all $v \in V(G) \setminus V(H)$. Then there exists an even cycle C in $G - V(H)$ such that both $G - V(C)$ and $G - E(C)$ are connected.*

Theorem 1.4, improves a result of canlon [3] which states that every connected graph of minimum degree at least 4 contains a non-separating even cycle. A result for non-separating odd cycles is obtained as follows:

Theorem 1.5. *Let G be a 3-connected graph and let H be a connected subgraph of G such that $G - V(H)$ contains an odd cycle. Then there exist an induced odd cycle C in $G - V(H)$ such that $G - V(C)$ is connected.*

It is proved by Thomassen and Toft [7] every 2-connected graph of minimum degree at least 4 contains a cycle C such that $G - V(C)$ and $G - E(C)$ is 2-connected. Borse [2] strengthen this result.

Theorem 1.6. *Let G be a 2-connected graph of minimum degree at least 4 which is not isomorphic to K_5 . Then there exist vertex disjoint cycles C_1 and C_2 such that $G - V(C_i)$ is connected and $G - E(C_i)$ is 2-connected for $i = 1, 2$.*

It follows from Theorem 1.6, that the 2-connected graph with minimum degree at least 4 which is not isomorphic to K_5 has two vertex disjoint cycles which are edge removable with respect to 2-connected graph. Further, those cycles are vertex removable with respect to connected graph.

Theorem 1.7. *Let G be a connected graph and let H be a connected subgraph of G such that $G - V(H)$ contains a cycle. If $d_G(v) \geq 3$ for all $v \in V(G) \setminus V(H)$ then there exists an induced cycle C in $G - V(H)$ such that $G - V(C)$ is connected.*

We need the following theorems:

Theorem 1.8. [8] *In complete graph with n -vertices there are $\frac{(n-1)}{2}$ edge-disjoint Hamiltonian cycles, if n is odd number ≥ 3 .*

Theorem 1.9. [4] *A strongly connected tournament T on n vertices contains directed cycle of length 3, 4, ..., n .*

Theorem 1.10. [5] *Every digraph in which $id, od v \geq \frac{n}{2}$ for all vertices v is Hamiltonian.*

2. Vertex Removable Cycles of Eulerian Graphs

In this section we characterized the eulerian graphs which contain a vertex removable cycles. The odd and even vertex removable cycles of eulerian graphs are characterized.

Theorem 2.1. *Let G be an eulerian graph with minimum degree at least 4. Then the cycle C in G is vertex removable if every vertex in G not in C which is adjacent to C is adjacent to even number of vertices in C .*

Proof. Let G be an eulerian graph with minimum degree at least 4. Then G is connected and every vertex in G has even degree. As G is connected, then by Thomassen and Toft theorem [7], G has a non-separating cycle C . That is G contains a cycle C such that $G - V(C)$ is connected.

Now, if every vertex in G not in C which is adjacent to C is adjacent to even number of vertices in C . Then removing the vertices of C reduce the degree of every vertex adjacent to C by even number, but the degree of every vertex in G is even, then the degree of every vertex in $G - V(C)$ is even. Hence $G - V(C)$ is eulerian and C is vertex removable cycle. ■

Canlon [3] proved that every connected graph with minimum degree at least 4 contains a non-separating even cycle, that is an even cycle which is vertex removable.

From canlon result we see that Theorem 2.1 is true when the vertex removable cycle is even, which characterize the vertex removable even cycles in eulerian graphs.

Here, we characterized eulerian graphs which contain vertex disjoint vertex removable cycles.

Theorem 2.2. *Let G be a simple eulerian graph of minimum degree at least 4, and G has two vertex disjoint cycles C_1 and C_2 . Then C_1 and C_2 are vertex removable if and only if every vertex not in C_1 which is adjacent to C_1 is adjacent to even number of vertices in C_1 , and every vertex not in C_2 which is adjacent to C_2 is adjacent to even number of vertices in C_2 , and G is not isomorphic to any of the following graphs: (i) a wheel; (ii) a graph with a partition (X, Y) of its vertex set such that $|X| = 3$ and $N(y) = X$ for all $y \in Y$.*

Proof. Let G be a simple eulerian graph of minimum degree at least 4, and G contains two vertex disjoint cycles C_1 and C_2 . Suppose that C_1 and C_2 are vertex removable. Then by Definition 1.1, both $G - V(C_1)$ and $G - V(C_2)$ are eulerian. Suppose that there is a vertex v in G not in C_1 such that v is adjacent to odd number of vertices in C_1 . Since G is eulerian, then G is connected and degree of every vertex in G is even. Thus v has even degree in G . Then removing $V(C_1)$ from G gives v with odd degree in C_1 . That is $G - V(C_1)$ has a vertex of odd degree. But $G - V(C_1)$ is eulerian a contradiction. Hence v is adjacent to even number of vertices in G and every vertex

in G not in C which is adjacent to C is adjacent to even number of vertices in G . Similarly, we prove that every vertex in G not in C_2 which is adjacent to C_2 is adjacent to even number of vertices in C_2 . It is clear that the wheel graph does not contain two vertex disjoint cycles, so G is not isomorphic to a wheel graph. Also a graph with a partition (X, Y) of its vertex set such that $|X| = 3$ and $N(y) = X$ for all $y \in Y$. does not contain two vertex disjoint cycles, therefore G is not isomorphic to this graph.

Conversely, suppose that C_1 and C_2 as in the hypothesis of the theorem and, G is not isomorphic to wheel graph, also not isomorphic to a graph with a partition (X, Y) of its vertex set such that $|X| = 3$ and $N(y) = X$ for all $y \in Y$. Since G is simple graph, then G has no loop and has no multiple edges. Thus G does not contains a cycle of length 1 or 2. Then every cycle in G has a length greater than or equal 3. As G has two vertex disjoint cycles, then $V(G)$ must be greater than or equal 6. Thus $|V(G)| \geq 6$. Then by Theorem 1.2, both $G - V(C_1)$ and $G - V(C_2)$ are connected. We prove that every vertex in $G - V(C_1)$ has even degree and every vertex in $G - V(C_2)$ has even degree. Since every vertex not in C_1 which is adjacent to C_1 is adjacent to even number of vertices in C_1 , then when we remove $V(C_1)$ from G the degree of every vertex in G which is adjacent to C_1 reduces by even degree. But G is eulerian, by other word every vertex in G has even degree. Hence every vertex in $G - V(C_1)$ has even degree. As we have $G - V(C_1)$ is connected. Thus $G - V(C_1)$ is eulerian and C_1 is vertex removable cycle. Similarly, we prove that $G - V(C_2)$ is eulerian and C_2 is vertex removable cycle which is disjoint from C_1 . ■

Now, we introduce the necessary condition for eulerian graph to have an odd vertex removable cycle.

Theorem 2.3. *Let G be a 3- connected eulerian graph and let H be a connected subgraph of G such that $G - V(H)$ contains an odd cycle. Then G contains an odd vertex removable cycle C , if every vertex in G not in C which is adjacent to C is adjacent to even number of vertices in C .*

Proof. Let G be a 3- connected eulerian graph and let H be a connected subgraph of G such that $G - V(H)$ contains an odd cycle. Then by Theorem 1.5, G contains an odd non-separating cycle C . That is G contains an odd cycle C such that $G - V(C)$ is connected. To prove that the degree of every vertex in $G - V(H)$ is even. If every vertex in G not in C which is adjacent to C is adjacent to even number of vertices, then removing $V(C)$ from G reduce the degree of every vertex in G which is adjacent to C by even number, but the degree of every vertex in G is even, then every vertex in $G - V(C)$ has even degree. Hence $G - V(C)$ is eulerian and C is odd vertex removable cycle.

Here we characterized the eulerian graphs which contain two vertex disjoint removable cycles. ■

Theorem 2.4. *Let G be a 2-connected eulerian graph of minimum degree at least 4 which is not isomorphic to K_5 . Then G contains two vertex disjoint cycles C_1 and C_2*

such that

- (i) C_1 and C_2 are edge removable with respect to eulerian graphs.
- (ii) C_1 and C_2 are vertex removable with respect to eulerian graphs if every vertex not in C_1 which is adjacent to C_1 is adjacent to even number of vertices in C_1 , and every vertex not in C_2 which is adjacent to C_2 is adjacent to even number of vertices in C_2 .
- (iii) C_1 and C_2 are edge removable with respect to 2-connected graph.

Proof. Let G be a 2-connected eulerian graph of minimum degree at least 4 which is not isomorphic to K_5 . Then by Theorem 1.6, there exist vertex disjoint cycles C_1 and C_2 such that $G - V(C_i)$ is connected and $G - E(C_i)$ is 2-connected for $i = 1, 2$.

To prove that both C_1 and C_2 are edge removable with respect to eulerian graphs. It is sufficient to prove that every vertex in $G - E(C_i)$ has even degree for $i = 1, 2$. As G is eulerian, G is connected and every vertex in G has even degree. It is clear that removing the edges of any cycle C in G reduce the degree of every vertex in C by 2. But every vertex in G is even. Thus every vertex in $G - E(C_i)$ is even, for $i = 1, 2$.

Also, from Theorem 1.6, we have $G - E(C_i)$ is connected for $i = 1, 2$. Hence $G - E(C_i)$ is eulerian for $i = 1, 2$. Thus both C_1 and C_2 are vertex disjoint edge removable cycles with respect to eulerian graphs, and (i) holds.

To prove that both C_1 and C_2 are vertex removable with respect to eulerian graphs. From Theorem 1.6, $G - V(C_i)$ is connected for $i = 1, 2$. We prove that every vertex in $G - V(C_i)$ has even degree for $i = 1, 2$. If every vertex in G not in C which is adjacent to C_i is adjacent to even number of vertices in C_i , then removing the vertices of C_i from G reduce the degree of every vertex in G which is adjacent to C_i by even number. But the degree of every vertex in G is even. Thus every vertex in $G - V(C_i)$ has even degree for $i = 1, 2$. Hence $G - V(C_i)$ is eulerian for $i = 1, 2$. Thus C_1 and C_2 are vertex disjoint vertex removable cycles, and (ii) holds.

(iii) follows directly from Theorem 1.6. ■

We also get the following result regarding vertex removable cycles of eulerian graph.

Theorem 2.5. *Let G be an eulerian graph and let H be a connected subgraph of G such that $G - V(H)$ contains a cycle. If $d_G(v) > 3$ for all $v \in V(G)/V(H)$ then there exists an induced cycle C in $G - V(H)$ which is vertex removable with respect to eulerian graph if every vertex in G not in C which is adjacent to C is adjacent to even number of vertices in C .*

Proof. Suppose that G is eulerian graph and let H be a connected subgraph of G such that $G - V(H)$ contains a cycle. If $d_G(v) > 3$ for all $v \in V(G)/V(H)$. As G is eulerian, G is connected and every vertex in G has even degree. By Theorem 1.7, there exists an induced cycle C in $G - V(H)$ such that $G - V(c)$ is connected. We prove that every vertex in $G - V(c)$ has even degree.

If the vertices in G not in C which is adjacent to C is adjacent to even number of vertices, then it is clear removing the vertices of C reduce the degree of each vertex in G which is adjacent to C by even number. That is every vertex in $G - V(c)$ has even degree. Thus $G - V(c)$ is eulerian and C is vertex removable cycle. ■

3. Edge Removable Cycles of Regular Graphs

In this section we characterized the edge removable cycles of regular graphs.

In the following result Attar [1] characterized the regular graph which contains an edge removable cycle.

Theorem 3.1. *The connected r -regular graph G has an edge removable cycle if and only if it contains a Hamiltonian cycle.*

Theorem 3.2. *Let G be an r -regular graph, where r is a positive integer even number. Then either G contains one edge removable cycle or there exist k disjoint cycles C_1, C_2, \dots, C_k , in G such that $\bigcup_{i=1}^k C_i$ is edge removable.*

Here we obtain the sufficient condition for existence an edge removable cycle of regular graphs.

Theorem 3.3. *Let G be r -regular graph. Then G has an edge removable cycle if and only if G contains a 2-factor.*

Proof. Let G be r -regular graph, suppose that G has an edge removable cycle. Then G contains a cycle C such that $G - E(C)$ is regular. Suppose that there exists a vertex v in G such that v is not in C . As the cycle enters and exists each vertex one time, then removing $E(C)$ from G reduce the regularity degree of G by 2. Then the degree of every vertex in $G - E(C)$ is $r - 2$. Since v is not in C , the degree of v is r . But $G - E(C)$ is regular. Then we must have $r - 2 = r$ which is a contradiction. Hence v is in C , and every vertex in G is in C . Thus C is a spanning 2-regular subgraph. That is C is a 2-factor.

Conversely, suppose that G contains a 2-factor. That is G contains a spanning 2-regular subgraph H . It is clear that either H is isomorphic to a Hamiltonian cycle or H is isomorphic to a union of distinct cycles C_1, C_2, \dots, C_k . If H is isomorphic to a Hamiltonian cycle C , then G must be connected and by Theorem 3.1, C is edge removable cycle.

If H is isomorphic to a union of distinct cycles C_1, C_2, \dots, C_k . As H is a spanning 2-regular subgraph then every vertex in G is belong to exactly one cycle of C_1, C_2, \dots, C_k . Then removing the edges of the cycles C_1, C_2, \dots, C_k reduce the degree of every vertex in G by 2. That is the degree of every vertex in $G - E(\bigcup_{i=1}^k C_i)$ is $(r - 2)$. Hence $G - E(\bigcup_{i=1}^k C_i)$ is $(r - 2)$ -regular graph, and $\bigcup_{i=1}^k C_i$ are edge removable cycles. ■

Theorem 3.4. *Let G be $(n - 1)$ -regular graph on odd order $n \geq 3$, then G contains $\frac{(n-1)}{2}$ edge- disjoint edge removable cycles.*

Proof. Suppose that G is regular graph on odd order $n \geq 3$, by Theorem 1.8, G contains $\frac{(n-1)}{2}$ edge- disjoint Hamiltonian cycles. Let C be one of these Hamiltonian cycles in G . As the cycle enters and exists each vertex one time. We can see easily removing the edge of C from G reduce regularity degree of each vertex in G by 2. That is every

vertex in $G - E(C)$ has a degree $(n - 3)$. Hence $G - E(C)$ is $(r - 3)$ -regular graph, and C is edge removable cycle in G . Similarly for the other cycles. ■

4. Edge Removable Cycles of Regular Digraphs

In this section we introduced some results for regular digraphs which contain an edge removable direct cycles.

Theorem 4.1. *Let D be a strongly connected r -regular digraph on n vertices such that the underlying graph of D is complete graph, then D has an edge removable cycle of length n with respect to regular digraph.*

Proof. Suppose that D is a strongly connected r -regular digraph on n vertices such that the underlying graph of D is complete graph. Then D is a tournament. By Theorem 1.9, D contains a directed cycles of length $3, 4, \dots, n$. Consider C as a directed cycle of length n in D . It is clear that C is Hamiltonian. Then removing $E(C)$ from D reduce the indegree and the outdegree of every vertex in D by 1. That is $D - E(C)$ is $(r - 1)$ -regular digraph. Hence C is an edge removable directed cycle of length n . ■

Theorem 4.2. *Every regular digraph in which $id, od(v) \geq \frac{r}{2}$ for all v has an edge removable directed cycle.*

Proof. The proof follows from Theorem 1.10, and the fact, removing the edges of Hamiltonian directed cycle from a regular digraph D preserve the regularity of D . ■

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